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"Research on Complex Function Theory"

by

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C O N T E N T S

ABSTRACT

The Nevanlinna characteristic of the derivative of a meromorphic function quasi-bounded in the unit circle is studied, and also a contribution is made towards the elucidation of the comparison of the characteristic of an entire function with its maximum modulus. Solutions are given to two of the problems raised by members of the Colloquium on Classical Function Theory at Cornell University, 1961. Finally proofs are given of a theorem on lacunary Fourier series, improving a result due to Tomić, and of a theorem on rearrangement of series, related to a result due to Agnew.

PUBLICATIONS

Arising out of the work done under the grant, the following articles have been accepted for publication:

"A problem on bounded analytic functions"
by P. B. Kennedy (Proc. Amer. Math. Soc.);

"Note on Fourier series with Hadamard gaps"
by P. B. Kennedy (Journal London Math. Soc.);

and the following article has been submitted for publication:

"On the derivative of a quasi-bounded function"
by P. B. Kennedy (Quart. J. Math. Oxford).

CHAPTER ONE

§ 1.1. Let $f(z)$ be meromorphic for $|z| < 1$, and for $0 < r < 1$ define

$$m(r,f) = (2\pi)^{-1} \int_0^{2\pi} \log^+ |f(re^{i\alpha})| d\alpha,$$

$$N(r,f) = \int_0^r (n(t,f) - n(0,f))\, dt/t + n(0,f) \log r,$$

where $n(t,f)$ is the number of poles of f in $|z| \leq t$, each pole counted a number of times equal to its multiplicity. The function

$$T(r,f) = m(r,f) + N(r,f)$$

is called the Nevanlinna characteristic of f. Our object is to discuss the rate of growth of $T(r,f')$ subject to the restriction

$$T(r,f) = O(1) \text{ as } r \text{ tends to } 1. \qquad (1.1.1)$$

Functions f satisfying (1.1.1) are called quasi-bounded. Every bounded regular function is quasi-bounded, but of course the converse is false in general: in fact (bibl. 8, page 133) f is quasi-bounded if, and only if, f is the quotient of two bounded regular functions. The question, whether the derivative of a quasi-bounded function is itself quasi-bounded, was raised by Nevanlinna (bibl. 8, page 138), who attributed it to Bloch. Many functions are now known which answer this question in the negative(even for bounded regular f). The earliest example is due to Frostman (bibl. 4; for other examples see 7, 11). If f is bounded (and so regular), it is elementary that

$$f'(z) = O(1)/(1 - |z|),$$

so that

$$T(r,f') = m(r,f') < -\log(1-r) + O(1);$$

and I have shown (bibl. 7, Theorem IV) that, if $\beta(r)$ tends to infinity as r tends to 1, then there is a bounded regular f such that

$$T(r,f') > -\log(1-r) - \beta(r) \qquad (1.1.2)$$

on a <u>sequence</u> of r tending to 1. Questions which now arise naturally are: does (1.1.1) imply

$$T(r,f') + \log(1-r) \longrightarrow -\infty ? \qquad (1.1.3)$$

and if so, then, given $\beta(r)$ tending to infinity as r tends to 1, can (1.1.2) hold for some quasi-bounded f and <u>all sufficiently small</u> $1 - r$? A question closely related to the second of these is the following, proposed by members of the Colloquium on Classical Function Theory at Cornell University, 1961 (bibl. 3, problem 5):

"Is there a bounded analytic function defined in $|z| < 1$ such that

$$N(r,1/f')/(-\log(1-r)) \longrightarrow 1 \text{ as } r \longrightarrow 1 ? \qquad (1.1.4)$$

Can such an example be constructed as a gap series

$$\sum c_k z^{n_k}, \qquad \sum |c_k| < \infty ?\text{"}$$

In this chapter I answer these and some similar questions.

§ 1.2. In fact (1.1.1) does imply (1.1.3). This is a consequence of Theorem I, proved below; but it seems of interest to give first a proof involving the use of Fatou's theorem. If α is real denote by $S(\alpha)$ the "Stolz region" which is the part of $|z| < 1$ where

$$|\arg(1 - ze^{-i\alpha})| \leq \pi/4.$$

We then have the following well-known form of Fatou's theorem.

LEMMA 1. <u>Let</u> $w(z)$ <u>be regular and bounded in</u> $|z| < 1$. <u>Then</u> $\lim w(z)$ <u>exists for almost all</u> α, <u>as</u> z <u>tends to</u> $e^{i\alpha}$ <u>in</u> $S(\alpha)$; <u>and for fixed</u> α <u>the convergence is uniform with</u>

respect to arg z *as* z *tends to* $e^{i\alpha}$ *in* $S(\alpha)$.

We deduce

LEMMA 2. *With the hypothesis of Lemma 1,* $(1 - r)\, w'(re^{i\alpha})$ *tends to* 0 *boundedly for almost all* α.

To prove Lemma 2, fix any α such that $L(\alpha)$, the limit of $w(z)$ as z tends to $e^{i\alpha}$ in $S(\alpha)$, exists. It is plain that there is an absolute constant $\delta > 0$ such that, as soon as $r > \frac{1}{2}$, $S(\alpha)$ contains the closed disc with centre $re^{i\alpha}$ and radius $\delta(1 - r)$. Let $\varepsilon > 0$ be given; then we may choose $r_o > \frac{1}{2}$ so that $|w(z) - L(\alpha)| < \varepsilon$ for all z satisfying

$$|z - re^{i\alpha}| < \delta(1 - r), \quad r_o < r < 1.$$

Hence for $r_o < r < 1$, by Cauchy's inequality for $w'(z)$,

$$|w'(re^{i\alpha})| = \left|(d/dz)(w(z)-L(\alpha))\right|_{z=re^{i\alpha}} \leq \varepsilon/(\delta(1-r)).$$

Because $\delta > 0$ is an absolute constant and $\varepsilon > 0$ is arbitrary, it follows that $w'(re^{i\alpha}) = o(1)/(1 - r)$; and this is true for almost all α because $L(\alpha)$ exists for almost all α by Lemma 1. Moreover since w is bounded we have $w'(re^{i\alpha}) = O(1)/(1 - r)$ uniformly with respect to α. This proves Lemma 2.

We note also the following inequality (bibl. 14, page 207, footnote).

LEMMA 3. *Let* $u(x)$ *be a positive function integrable over* (a,b). *Then*

$$(b-a)^{-1} \int_a^b \log u(x)\, dx \leq \log\left((b-a)^{-1} \int_a^b u(x)\, dx\right).$$

Suppose now that f is quasi-bounded, so that $f = g/h$, where g and h are bounded in $|z| < 1$. Since

$$f' = (g'h - gh')/h^2, \tag{1.2.1}$$

we have easily

$$|f'| \leq (|g'| + |h'|)\,(|g| + |h|)/|h^2|,$$

and so, because g/h^2 and $1/h$ are quasi-bounded,

$$m(r,f') < (2\pi)^{-1} \int_0^{2\pi} \log\,(1+|g'(re^{i\alpha})|+|h'(re^{i\alpha})|)\,d\alpha + O(1).$$

Hence by Lemma 3 there is a constant A such that

$$\exp m(r,f') < A \int_0^{2\pi} (1 + |g'(re^{i\alpha})| + |h'(re^{i\alpha})|)\,d\alpha.$$

By Lemma 2 and the theorem of bounded convergence (bibl. 12, page 337) the integral on the right is $o(1)/(1 - r)$, and so $m(r,f') + \log(1-r)$ tends to $-\infty$. But by (1.1.1) and (1.2.1),

$$N(r,f') \leqslant 2N(r,f) \leqslant 2T(r,f) = O(1),$$

and so $T(r,f') < m(r,f') + O(1)$. This proves (1.1.3).

§ 1.3. Although it seemed worthwhile to present the argument of § 1.2, there is in fact a simpler and yet stronger argument.

THEOREM I. <u>Let</u> f <u>be quasi-bounded in</u> $|z| < 1$. <u>Then</u>

$$\int_0^1 (1 - r)\exp 2T(r,f')\,dr < \infty. \qquad (1.3.1)$$

<u>Proof</u>. As before, $f = g/h$ where g and h are bounded regular functions. By (1.2.1)

$$|f'| \leqslant ((|g|+|h|)/|h^2|)\max(|g'|, |h'|),$$

and so

$$|f'|^2 \leqslant (|g'|^2 + |h'|^2)(|g| + |h|)^2/|h|^4.$$

Since g^2/h^4, g/h^3 and $1/h^2$ are quasi-bounded, it follows that $2T(r,f') < 2m(r,f') + O(1)$

$$< (2\pi)^{-1} \int_0^{2\pi} \log\,(1 + |g'(re^{i\alpha})|^2 + |h'(re^{i\alpha})|^2)\,d\alpha + O(1).$$

Employing Lemma 3 again we find that there are constants A, A' such that

$$\exp 2T(r,f') < A \int_0^{2\pi} (1+|g'(re^{i\alpha})|^2 + |h'(re^{i\alpha})|^2)\, d\alpha$$

$$= A'\left(1 + \sum_1^{\infty} n^2(|g_n^2| + |h_n^2|)\, r^{2n-2}\right)$$

where g_n, h_n are respectively the nth Taylor coefficients of g, h at z = 0. If we now multiply by 1 - r and integrate from 0 to 1 we get (1.3.1), since

$$\sum |g_n^2| < \infty, \qquad \sum |h_n^2| < \infty$$

because g and h are bounded. This proves Theorem I.

From Theorem I we deduce (1.1.3) as an easy corollary. In fact, let $\varepsilon > 0$ be given; then for all small enough 1 - r, from (1.3.1) and the fact that T increases with r (bibl. 8, page 7), we get

$$\varepsilon > \int_r^{\frac{1}{2}(1+r)} (1 - t) \exp 2T(t,f')\, dt$$

$$> (1/4)\,(1 - r)^2 \exp 2T(r,f'),$$

and (1.1.3) follows at once from this.

§ 1.4. It is plain from Theorem I that, for an arbitrary $\beta(r)$ tending to infinity, we cannot have (1.1.2) for some quasi-bounded f and all small enough 1 - r: for example, if $\beta(r) = \frac{1}{2}\log(-\log(1-r))$, then (1.1.2) (for all sufficiently small 1 - r) would contradict (1.3.1). Our object now is to show that Theorem I is, in fact, in a sense best-possible.

THEOREM II. *Let* B(r) *be positive and increasing in* (0,1), *and satisfy*

$(1 - r) \exp B(r)$ decreases as r increases, (1.4.1)

$B(r') - B(r'') \longrightarrow \infty$ as $(1-r')/(1-r'') \longrightarrow 0$, (1.4.2)

$$\int_0^1 (1 - r) \exp 2B(r)\, dr < \infty. \tag{1.4.3}$$

Then there exists a function f, regular and quasi-bounded in $|z| < 1$, such that

$$T(r,f') = m(r,f') > B(r) \tag{1.4.4}$$

for all sufficiently small $1 - r$.

The class of functions B considered is fairly wide; we note, for example, that (1.4.1) and (1.4.2) are satisfied if B is continuously differentiable and satisfies, for some constant $c > 0$,

$$c/(1 - r) < dB(r)/dr < 1/(1 - r).$$

To prove Theorem II we observe first that, if the range of integration is $(1 - 2^{-k}, 1 - 2^{-k-1})$,

$$\int (1-r) \exp 2B(r)\, dr > 2^{-2k-2} \exp 2B(1 - 2^{-k});$$

here we have used the facts that $B(r)$ increases with r and that the length of the range of integration is 2^{-k-1}. By summing over $k = 0, 1, 2, \ldots$ and using (1.4.3) we get

$$\sum 2^{-2k} \exp 2B(1 - 2^{-k}) < \infty.$$

It is easy to infer from this that we may then choose $\gamma(k)$, a function taking only integer values greater than 2 and increasing to infinity as k tends to infinity, so that

$$\sum \gamma(k)^2\, 2^{-2k} \exp 2B(1 - 2^{-k}) < \infty, \tag{1.4.5}$$

and

$$\gamma(k+1)/\gamma(k) \text{ tends to } 1. \tag{1.4.6}$$

Define the sequence of integers n_k by

$$n_1 = 1, \quad n_{k+1} = \gamma(k) n_k \quad (\text{all } k > 0). \tag{1.4.7}$$

In particular $n_{k+1} > 2^k$, and so by (1.4.1)

$$n_{k+1}^{-1} \exp B(1 - n_{k+1}^{-1}) < 2^{-k} \exp B(1 - 2^{-k}).$$

If we multiply by $\gamma(k)$, square and use (1.4.5) and (1.4.7) we get

$$\sum n_k^{-2} \exp 2B(1 - n_{k+1}^{-1}) < \infty .$$

Now put

$$f(z) = \sum_{1}^{\infty} c_k z^{n_k} \tag{1.4.8}$$

where $c_k = e^2 n_k^{-1} \exp B(1 - n_{k+1}^{-1})$. Since

$$\sum c_k^2 < \infty,$$

f is regular in the unit circle and it is easy to deduce from Lemma 3 that f is quasi-bounded.

The proof of (1.4.4) depends on the following lemma, which is very easy to prove.

LEMMA 4. <u>If</u> s_k <u>is positive for all</u> k <u>and</u> s_k/s_{k+1} <u>tends to</u> 0 <u>as k tends to infinity, then</u>

$$\sum_{1}^{k-1} s_j = o(s_k), \qquad \sum_{k+1}^{\infty} s_j^{-1} = o(s_k^{-1}).$$

Lemma 4 gives at once

$$\sum_{1}^{k-1} c_j n_j = o(c_k n_k), \tag{1.4.9}$$

because

$$(c_k n_k)/(c_{k+1} n_{k+1}) = \exp\left(B(1-n_{k+1}^{-1}) - B(1-n_{k+2}^{-1})\right),$$

which tends to 0 by (1.4.2) and (1.4.7). Also

$$\sum_{j=k+1}^{\infty} c_j n_j (1 - n_k^{-1})^{n_j} < 2n_k^2 \sum_{k+1}^{\infty} c_j/n_j \tag{1.4.10}$$

by the inequality $(1 - x)^n < e^{-nx} < 2(nx)^{-2}$, which holds for $n > 0$ and $0 < x < 1$. But

$$(n_k/c_k) \,/\, (n_{k+1}/c_{k+1})$$

$$= \gamma(k)^{-2} \exp\left(B(1-n_{k+2}^{-1}) - B(1-n_{k+1}^{-1})\right),$$

and by (1.4.1) and (1.4.7) it follows that

$$(n_k/c_k) \,/\, (n_{k+1}/c_{k+1}) \leqslant \gamma(k+1)/\gamma(k)^2,$$

which tends to 0 by (1.4.6). Therefore by (1.4.10) and Lemma 4,

$$\sum_{k+1}^{\infty} c_j n_j (1 - n_k^{-1})^{n_j} = o(c_k n_k). \qquad (1.4.11)$$

Now

$$|f'(z)| \geqslant \left|\sum_{1}^{\infty} c_j n_j \, z^{n_j}\right|$$

and for z on the circle with centre 0 and radius $1-n_k^{-1}$, (1.4.9) and (1.4.11) tell us that this series is essentially dominated by the term

$$c_k n_k (1 - n_k^{-1})^{n_k},$$

so that on this circle

$$|f'(z)| > (e^{-1} - o(1))\, c_k n_k - o(c_k n_k).$$

Hence if k is large enough we have, by definition of c_k,

$$m(1 - n_k^{-1}, f') > B(1 - n_{k+1}^{-1}).$$

Since m and B are increasing functions of r we have then, for $1 - n_k^{-1} < r < 1 - n_{k+1}^{-1}$,

$$m(r,f') > m(1-n_k^{-1},f') > B(1-n_{k+1}^{-1}) > B(r).$$

It follows that (1.4.4) is true for all small enough $1 - r$, and this proves Theorem II.

§ 1.5. Theorem II shows Theorem I to be sharp for <u>quasi-bounded</u> functions, but perhaps not for <u>bounded</u> functions. It seems of some interest that the construction used in Theorem II, depending as it does on lacunary power series, is essentially incapable of showing Theorem I best-possible for bounded f. In fact, for bounded functions with lacunary Taylor series round z=0, we have the following stronger result.

THEOREM III. <u>Suppose that the sequence of positive integers</u> n_k <u>satisfies, as</u> k <u>tends to infinity,</u>

$$\liminf \; n_{k+1}/n_k \; > \; 1, \tag{1.5.1}$$

<u>and that the function (1.4.8) is regular and bounded in the unit</u> z-<u>circle. Then</u>

$$\int_0^1 \exp T(r,f') \, dr \; < \; \infty . \tag{1.5.2}$$

To prove Theorem III we observe that the real and imaginary parts of

$$\sum c_k e^{in_k\alpha}$$

are Fourier series of bounded functions of α (bibl. 16, page 86), and so by a theorem of Szidon (bibl. 16, page 139), (1.5.1) implies

$$\sum |c_k| \; < \; \infty . \tag{1.5.3}$$

(1.5.2) follows from (1.5.3) by a simple argument involving the use of Lemma 3, and this proves Theorem III.

(1.5.2) is stronger than (1.3.1) of course; for example (1.5.2), but not (1.3.1), would be contradicted if we had $T(r,f') > -\log(1-r) - \log(-\log(1-r))$ for all small enough $1 - r$. There is an analogue of Theorem II, as follows.

THEOREM IV. <u>Let</u> $B(r)$ <u>be positive and increasing in</u> $(0,1)$, <u>and satisfy (1.4.1), (1.4.2) and</u>

$$\int_0^1 \exp B(r) \, dr \; < \; \infty . \tag{1.5.4}$$

<u>Then there exists a function</u> f, <u>regular and bounded in the unit</u> z-<u>circle, having a Taylor series of the form (1.4.8) with</u> n_{k+1}/n_k <u>tending to infinity, and satisfying (1.4.4) for all sufficiently small</u> $1 - r$.

The proof is very similar to that of Theorem II. We easily show from (1.5.4) that

$$\sum 2^{-k} \exp B(1 - 2^{-k}) < \infty,$$

and so we may choose $\gamma(k)$ as in §1.4, taking only integer values greater than 2 and tending to infinity with k, to satisfy

$$\sum \gamma(k)\, 2^{-k} \exp B(1 - 2^{-k}) < \infty$$

and (1.4.6). Define n_k, c_k and $f(z)$ as in §1.4. Then $\sum c_k < \infty$ and so f is bounded; and by (1.4.7) n_{k+1}/n_k tends to infinity with k. The proof of (1.4.4) proceeds as in §1.4 with minor differences, and this proves Theorem IV.

It is now natural to ask: is (1.5.2) true for all bounded f, even without the gap hypothesis (1.5.1)? The methods used here are insufficient to answer this, but I hazard the conjecture that the answer is "no".

§ 1.6. The last question referred to in §1.1 is, in effect, answered in the affirmative by Theorem IV. We look more closely at this special case in the following theorem.

THEOREM V. <u>Suppose that</u>

$$|c_k| / |c_{k+1}| \text{ tends to } 1, \tag{1.6.1}$$

$$\sum |c_k| < \infty, \tag{1.6.2}$$

<u>and that the sequence of positive integers</u> n_k <u>satisfies</u>

$$n_{k+1}/n_k \text{ tends to infinity}, \tag{1.6.3}$$

$$(\log n_{k+1}) / (\log n_k) \text{ tends to } 1. \tag{1.6.4}$$

<u>Define</u> f <u>by (1.4.8). Then</u> f <u>is regular and bounded in the unit</u> z-<u>circle, and (1.1.4) is true</u>.

<u>On the other hand, suppose that</u> c_k <u>satisfies only</u>

$$c_k = O(1)\, k^p \qquad (1.6.5)$$

<u>for some fixed</u> p, <u>that (1.5.1) holds and that (1.6.4) is false. Define</u> f <u>by (1.4.8). Then (1.1.4) is false</u>.

The point of the second part is that it shows, for lacunary series, the essential part played by (1.6.4) in producing the behaviour (1.1.4).

Theorem V is proved by estimates very similar to those in the proofs of Theorems II and IV. To prove the first part we note first that f is a bounded regular function by (1.6.2). Put

$$r_k = \exp(-1/n_k). \qquad (1.6.6)$$

Then for all k and all real α,

$$|f'(r_k e^{i\alpha})| \geqslant e^{-1}|c_k| n_k - S_1 - S_2 ,$$

where

$$S_1 = \sum_{1}^{k-1} |c_j| n_j , \quad S_2 = \sum_{k+1}^{\infty} |c_j| n_j\, r_k^{n_j}.$$

From (1.6.1), (1.6.3), (1.6.6) and Lemma 4 the estimates

$$S_1,\ S_2 = o(|c_k| n_k)$$

follow easily. Moreover from (1.6.1) we get $\log c_k = O(k)$, and $k = o(\log n_k)$ from (1.6.3), so that

$$\log |c_k| = o(\log n_k).$$

Thus

$$\log |f'(r_k e^{i\alpha})| > (1 - o(1)) \log n_k$$

uniformly with respect to α; and noting that

$$f'(r_k e^{i\alpha}) = O(1)/(1 - r_k) = O(n_k)$$

by (1.6.6) and the fact that f is bounded, we get

$$\log |f'(r_k e^{i\alpha})| = (1 + o(1)) \log n_k$$

uniformly with respect to α. But (bibl. 12, page 249)

$$N(r,1/f') = (2\pi)^{-1} \int_0^{2\pi} \log |f'(re^{ia})| \, da + O(1),$$

and so

$$N(r_k,1/f') = (1 + o(1)) \log n_k.$$

From this (1.1.4) follows, because N is an increasing function of r and because the expressions

$$\log n_k, \quad - \log (1-r_k), \quad - \log (1-r_{k+1})$$

are all asymptotic to one another as k tends to infinity, by (1.6.4) and (1.6.6). This proves the first part of Theorem V.

To prove the second part we suppose that, in contradiction of (1.6.4), there exist a fixed $\delta > 1$ and infinitely many k such that

$$n_{k+1} > n_k^{\delta}. \qquad (1.6.7)$$

Put $t_k = \exp(-1/n_k^{\beta})$ for some (temporarily fixed) β with $1 < \beta < \delta$. Then for z on the circle with centre 0 and radius t_k, we have by (1.6.5)

$$f'(z) = O(1) \left\{ \sum_{j=1}^{k} j^p n_j + \sum_{k+1}^{\infty} j^p n_j t_k^{n_j} \right\}.$$

Using (1.6.3) we can show that the first sum on the right is

$$O(1)\, n_k^{1+\varepsilon} \text{ for every } \varepsilon > 0,$$

and that the second sum is bounded for all k for which (1.6.7) is true. It is easy to conclude that, for every $\varepsilon > 0$ and all large enough k for which (1.6.7) is true,

$$N(t_k,1/f') < - \beta^{-1} (1 + \varepsilon) \log (1 - t_k).$$

Since β is also arbitrary in $(1,\delta)$ we deduce that

$$\liminf N(r,1/f')/(-\log (1 - r))$$

is at most $1/\delta$, which is less than 1. Thus (1.1.4) is false, and this completes the proof of Theorem V.

§ 1.7. To end this chapter we show that a method used in Theorem V suffices to show the accuracy of a result of Biernacki (bibl. 2, page 101). Biernacki shows that, <u>if the function</u> f, <u>regular in the unit</u> z-<u>circle, satisfies as</u> r <u>tends to</u> 1

$$\liminf T(r,f)/(-\log(1-r)) = c,$$

<u>then</u> $$\limsup T(r,f')/T(r,f) \leq 1 + 1/c$$

<u>as</u> r <u>tends to</u> 1 <u>outside a set of intervals over which the total variation of</u> $-\log(1-r)$ <u>is finite.</u> We prove

THEOREM VI. <u>Let</u> c <u>be given</u> (0, finite and positive, or ∞). <u>Then there exists</u> f, <u>regular in the unit</u> z-<u>circle and satisfying, as</u> r <u>tends to</u> 1,

$$T(r,f) \,/\, (-\log(1-r)) \text{ tends to } c,$$

$$T(r,f')/T(r,f) \text{ tends to } 1 + 1/c.$$

In view of what has gone before, the calculations involved in the proof of Theorem VI are straightforward and we omit them. For the case $c = 0$ we need only construct, as in Theorem II, a function f satisfying $T(r,f) = O(1)$, $T(r,f')$ tends to infinity. For finite $c > 0$ we may take

$$f(z) = \sum n_k^{\,c}\, z^{n_k}$$

for a sequence of integers n_k satisfying (1.6.3) and (1.6.4); while, with the same sequence n_k,

$$f(z) = \sum n_k^{\,k}\, z^{n_k}$$

deals with the case $c = \infty$. This proves Theorem VI.

CHAPTER TWO

§ 2.1. In this chapter $T(r,f)$ again denotes the Nevanlinna characteristic of f, but we are now concerned with functions meromorphic in the open plane, and especially with entire (integral) functions. For meromorphic f the <u>order</u> p of f is defined by

$$p \quad = \quad \limsup \, (\log T(r,f))/\log r$$

as r tends to infinity, which (bibl. 8, page 30) coincides with the usual definition of the order of f if f is entire. When f is entire we define, as usual, $M(r,f)$ to be the maximum modulus of $f(z)$ for z on the circle with centre 0 and radius r, and f is said to have <u>perfectly regular growth</u> if

$$\lim \; r^{-p} \log M(r,f) \tag{2.1.1}$$

(as r tends to infinity) exists, being finite and positive. Among the Cornell Colloquium problems (bibl. 3, problem 2) is the following: "if f is entire and (2.1.1) exists, does

$$\lim \; r^{-p} \, T(r,f) \tag{2.1.2}$$

exist?" In other words, if f has perfectly regular growth in the usual sense, does f also have perfectly regular growth in the Nevanlinna sense? It is plain that in considering this question we lose nothing by taking the value of (2.1.1) to be 1.

From the inequalities

$$(1-k)(1+k)^{-1} \log M(kr,f) \leq T(r,f) \leq \log^{+} M(r,f),$$

which are true for all k in $(0,1)$ (bibl. 8, page 24), we deduce easily that, if

CHAPTER TWO

§ 2.1. In this chapter $T(r,f)$ again denotes the Nevanlinna characteristic of f, but we are now concerned with functions meromorphic in the open plane, and especially with entire (integral) functions. For meromorphic f the <u>order</u> p of f is defined by

$$p \quad = \quad \limsup\ (\log T(r,f))/\log r$$

as r tends to infinity, which (bibl. 8, page 30) coincides with the usual definition of the order of f if f is entire. When f is entire we define, as usual, $M(r,f)$ to be the maximum modulus of $f(z)$ for z on the circle with centre 0 and radius r, and f is said to have <u>perfectly regular growth</u> if

$$\lim\ r^{-p} \log M(r,f) \tag{2.1.1}$$

(as r tends to infinity) exists, being finite and positive. Among the Cornell Colloquium problems (bibl. 3, problem 2) is the following: "if f is entire and (2.1.1) exists, does

$$\lim\ r^{-p}\, T(r,f) \tag{2.1.2}$$

exist?" In other words, if f has perfectly regular growth in the usual sense, does f also have perfectly regular growth in the Nevanlinna sense? It is plain that in considering this question we lose nothing by taking the value of (2.1.1) to be 1.

From the inequalities

$$(1-k)(1+k)^{-1} \log M(kr,f) \leqslant T(r,f) \leqslant \log^{+} M(r,f),$$

which are true for all k in $(0,1)$ (bibl. 8, page 24), we deduce easily that, if

$$\lim r^{-p} \log M(r,f) = 1 \qquad (2.1.3)$$

as r tends to infinity, then

$$A(p) \leqslant \liminf r^{-p} T(r,f)$$

$$\leqslant \limsup r^{-p} T(r,f) \leqslant 1 \qquad (2.1.4)$$

where

$$A(p) = \max \left((k^p - k^{p+1})/(1+k) \right) \text{ for } 0 < k < 1.$$

By considering the value $k = \exp(-1/p)$ we find that

$$A(p) > (2e(1+p))^{-1}$$

for all $p > 0$; and more detailed calculations show that

$$\lim A(p) = 1 \text{ as } p \text{ tends to } 0, \qquad (2.1.5)$$

$$\lim 2ep\, A(p) = 1 \text{ as } p \text{ tends to infinity.} \qquad (2.1.6)$$

In this chapter I prove the following theorem.

THEOREM VII. *Suppose that* $0 < p < \infty$. *Then there exists an entire function* f *satisfying* (2.1.3) *and, as* r *tends to infinity*,

$$\limsup r^{-p} T(r,f) = 1, \qquad (2.1.7)$$

$$\liminf r^{-p} T(r,f) \leqslant \gamma(p), \qquad (2.1.8)$$

where

$$\gamma(p) = (\pi p)^{-1} \sin \pi p \quad (0 < p < \tfrac{1}{2}),$$

$$\gamma(p) = (\pi p)^{-1} \quad (p \geqslant \tfrac{1}{2}).$$

Since $\gamma(p) < 1$ this answers "no" in general to the question raised above; and, with (2.1.5) and (2.1.6), it also shows that the lower bound for $r^{-p} T(r,f)$ in (2.1.4) is roughly of the correct order as regards its dependence on p.

§ 2.2. To prove Theorem VII we need two lemmas.

LEMMA 5. *Suppose that* $0 < p < \infty$, *and let* D *denote the angular domain*

$-\delta < \arg z < \delta = \min(\pi, \pi/(2p))$.

Then there exists an entire function $g(z)$ satisfying

$$|g(z)| < 1 \text{ for all } z \text{ outside } D, \text{ if } p>\tfrac{1}{2} \text{ or } p=\tfrac{1}{2}, \quad (2.2.1)$$

$$\log |g(r)| = r^p + O(\log r) \quad (2.2.2)$$

as r tends to infinity, and

$$\log |g(re^{i\alpha})| < r^p \cos p\alpha \quad (2.2.3)$$

for all $re^{i\alpha}$ in D with $r > 2$.

Lemma 5 is, in effect, known. When p is at least $\frac{1}{2}$ the lemma is a special case of a theorem of the present writer (bibl. 5, Theorem II); we note that (2.2.3), though not included in the statement of that theorem, is an easy consequence of Lemma 11 of (bibl.5). When $0 < p < \frac{1}{2}$ we make use of the function

$$W(z) = \prod_{1}^{\infty} (1 + z/n^{1/p}).$$

For this function Wiman (bibl. 15) obtained (even for $0 < p < 1$) the asymptotic statements, as r tends to ∞,

$$\log W(r) = \pi \operatorname{cosec} \pi p.\ r^p + O(\log r),$$

$$\log |W(re^{i\alpha})| < \pi \operatorname{cosec} \pi p.\ r^p \cos p\alpha + O(\log r)$$

uniformly with respect to α. It is therefore easy to see that, if $\beta > 0$ satisfies

$$\pi \beta^p \operatorname{cosec} \pi p = 1,$$

then a positive integer N may be chosen so that

$$g(z) = \prod_{N}^{\infty} (1 + \beta z/n^{1/p})$$

has the properties (2.2.2) and (2.2.3).

LEMMA 6. If $0 < p < \infty$, then there exists an entire function $h(z)$ satisfying, uniformly with respect to α,

$$|h(re^{i\alpha})| < (2 + o(1)) \exp(r^p) \quad (2.2.4)$$

<u>as</u> r <u>tends to infinity, while there are sequences</u> r_n <u>and</u> t_n <u>tending to infinity with</u> n <u>such that</u>

$$|h(r_n e^{i\alpha})| > (1 - o(1)) \exp (r_n{}^p), \tag{2.2.5}$$

$$|h(t_n e^{i\alpha})| < \exp (o(t_n{}^p)), \tag{2.2.6}$$

<u>as</u> n <u>tends to infinity</u>.

For the proof of Lemma 6 it is convenient to introduce temporarily the notations

$$q = 1/p, \tag{2.2.7}$$

$$s_n = n! = n(n-1)(n-2)\ldots 3.2.1. \tag{2.2.8}$$

Put

$$h(z) = \sum_{n=1}^{\infty} (e^q p^q z/s_n{}^q)^{s_n}$$

$$= \sum_{n=1}^{\infty} u_n(z), \quad \text{say.}$$

When

$$(qs_{n-1})^q \leq r \leq (qs_n)^q \tag{2.2.9}$$

the power series is essentially dominated, on the circumference with centre 0 and radius r, by the two terms $u_{n-1}(z) + u_n(z)$. To see this we note first that if r, the modulus of z, satisfies (2.2.9), then by (2.2.7) and (2.2.8)

$$\left| \sum_{n+1}^{\infty} u_j(z) \right| \leq \sum_{j=n+1}^{\infty} (es_n/s_j)^{qs_j} < \sum_{j=n+1}^{\infty} (e/n)^{qs_j}$$

and so

$$\sum_{n+1}^{\infty} u_j(z) = o(1). \tag{2.2.10}$$

Further, because $(e^q p^q r/m^q)^m$ increases with m as long as $m^q < p^q r$, we have, when $|z| = r$ satisfies (2.2.9),

$$\left| \sum_{1}^{n-2} u_j(z) \right| < n(e^q p^q r/s_{n-2}{}^q)^{s_{n-2}} ,$$

and this is less than

$$n\,(en^2)^{qs_{n-2}}$$

because

$$p^q r/s_{n-2}{}^q \;<\; n^{2q}$$

by (2.2.7), (2.2.8) and the right-hand inequality of (2.2.9). Again by the left-hand inequality of (2.2.9),

$$qs_{n-2} \;\leqslant\; r^p/(n-1),$$

and so

$$\left|\sum_1^{n-2} u_j(z)\right| \;<\; n\,(en^2)^{r^p/(n-1)}$$

$$<\; n^{4r^p/(n-1)} \;=\; \exp\,(\,(4r^p \log n)/(n-1)\,).$$

Hence

$$\left|\sum_1^{n-2} u_j(z)\right| \;<\; \exp\,(o(r^p)). \qquad (2.2.11)$$

And now, when r, the modulus of z, satisfies (2.2.9), we get by (2.2.10) and (2.2.11)

$$|h(z) - u_{n-1}(z) - u_n(z)| \;<\; \exp\,(o(r^p)), \qquad (2.2.12)$$

which justifies our assertion regarding the essential domination by two terms of the power series defining h(z).

For all m we have

$$(e^q p^q r/m^q)^m \;\leqslant\; \exp\,(r^p),$$

and (2.2.4) follows at once from this and (2.2.12).

Put

$$r_n \;=\; (qs_n)^q.$$

Then $r = r_n$ satisfies (2.2.9), and by (2.2.7) and (2.2.8)

$$|u_n(r_n e^{i\alpha})| = \exp\,(qs_n) = \exp\,(r_n{}^p),$$

$$|u_{n-1}(r_n e^{i\alpha})| \;=\; u_{n-1}(r_n) \;=\; (e^q s_n{}^q/s_{n-1}{}^q)^{s_{n-1}}$$

$$= (en)^{qs_{n-1}} \;<\; n^{2r_n{}^p/n} \;=\; \exp\,(o(r_n{}^p)),$$

and so (2.2.5) follows easily from (2.2.12).

Similarly if we put

$$t_n = (qs_n)^q n^{-\frac{1}{2}q}$$

then $r = t_n$ satisfies (2.2.9), and for $|z| = t_n$

$$|u_{n-1}(z)| = u_{n-1}(t_n) < \exp (o(t_n{}^p)),$$

$$|u_n(z)| = u_n(t_n) = o(1).$$

Hence we get (2.2.6) from (2.2.12). This proves Lemma 6.

§ 2.3. We can now prove Theorem VII. Let g and h be as in Lemmas 5 and 6, and put

$$f(z) = z^Q g(z) + h(z), \tag{2.3.1}$$

where the positive integer Q is so chosen that

$$r^Q|g(r)| > 3 \exp (r^p) \quad (\text{all } r > 2);$$

such a choice of Q is possible by (2.2.2), and we then have, by (2.2.4) and (2.3.1),

$$|f(r)| > (1 - o(1)) \exp (r^p).$$

Moreover by (2.2.1), (2.2.3) and (2.2.4),

$$|f(re^{i\alpha})| < (r^Q + 2 + o(1)) \exp (r^p)$$

uniformly with respect to α. Hence (2.1.3) is true.

Next let δ be as in Lemma 5, take any fixed ε satisfying $0 < \varepsilon < \delta$, and suppose $\varepsilon < |\alpha| < \delta$. Then by (2.2.3)

$$|g(re^{i\alpha})| < \exp (r^p \cos p\varepsilon)$$

for all $r > 2$. Since $\cos p\varepsilon < 1$ we have therefore

$$(re^{i\alpha})^Q g(re^{i\alpha}) = o(1) \exp (r^p).$$

Hence if r_n is as in Lemma 6, we have by (2.2.5) and (2.3.1)

$$|f(r_n e^{i\alpha})| > (1 - o(1)) \exp (r_n^p).$$

If $p > \frac{1}{2}$ then this same inequality holds for $\delta < |\alpha| < \pi$ by (2.2.1), (2.2.5) and (2.3.1). Hence in any case

$$T(r_n,f) > (2\pi)^{-1}(2(\delta-\varepsilon) + 2(\pi-\delta)) r_n^p - o(1).$$

Since $\varepsilon > 0$ is arbitrarily small, (2.1.7) follows at once from this and the extreme right-hand inequality of (2.1.4).

Finally, with t_n as in Lemma 6,

$$T(t_n,h) = o(t_n^p)$$

by (2.2.6); and by (2.2.1) and (2.2.3)

$$T(t_n,g) < \gamma(p)\, t_n^p.$$

Hence (bibl. 8, page 14) we have by (2.3.1)

$$T(t_n,f) < T(t_n,g) + T(t_n,h) + O(\log t_n)$$

$$< \gamma(p)\, t_n^p + o(t_n^p).$$

This proves (2.1.8) and so completes the proof of Theorem VII.

§ 2.4. When I had finished the above work I was informed by Professor W. K. Hayman that I had been slightly anticipated by Professor A. A. Goldberg in answering "no" to the Cornell Colloquium question stated in §2.1. I have not an exact reference to Goldberg's paper. Instead of Lemma 5 he makes use of properties of the Mittag-Leffler function

$$\sum_{n=0}^{\infty} z^n/\Gamma(1 + n/p).$$

Moreover he uses a device different from the rather sharp Lemma 6, leading to a weaker result than (2.1.7). It should be mentioned that Goldberg also constructed an example to show that, conversely, the existence of (2.1.2) does not imply the existence of (2.1.1).

Professor Goldberg kindly called my attention, in a letter, to a paper of Paley (bibl. 10; for the exact reference I am indebted to Professor Hayman) where it is shown that there are entire functions f satisfying, as r tends to infinity,

$$\liminf \; T(r,f)/\log M(r,f) = 0,$$

and where it is also conjectured that, in general,

$$\limsup \; T(r,f)/\log M(r,f) \geq \gamma(p)$$

with $\gamma(p)$ as in Theorem VII. Paley states that, in fact, this conjecture is known to be true for $p < \frac{1}{2}$ and for $p = \frac{1}{2}$; the case $p > \frac{1}{2}$ remains open.

CHAPTER THREE

§ 3.1. In this chapter I turn my attention to two problems about infinite series.

Let the L-integrable function $f(x)$, with period 2π, have the Fourier series

$$\sum_{k=1}^{\infty} (a_k \cos n_k x + b_k \sin n_k x) \tag{3.1.1}$$

where the sequence of positive integers n_k satisfies

$$\liminf n_{k+1}/n_k > 1 \tag{3.1.2}$$

as k tends to infinity. Suppose that, for some fixed x_0 and $0 < \alpha \leq 1$,

$$f(x_0 + t) - f(x_0) = O(1)|t|^{\alpha} \tag{3.1.3}$$

as t tends to zero. Tomić (bibl. 13) has shown that we then have

$$a_k, \ b_k = O(1)\ n_k^{-\alpha/(2+\alpha)}. \tag{3.1.4}$$

I have previously (bibl. 6) shown that the hypothesis (3.1.3), uniformly for x_0 on a set of positive measure, is enough to ensure the stronger conclusion

$$a_k, \ b_k = O(1)\ n_k^{-\alpha};$$

and I show here that Tomić's result can be improved to the following.

THEOREM VIII. *Let the L-integrable function $f(x)$, with period 2π, have the Fourier series (3.1.1) satisfying (3.1.2), and suppose that (3.1.3) holds for some x_0 and $\alpha > 0$. Then*

$$a_k, \ b_k = O(1)\ (n_k^{-1} \log n_k)^{\alpha} \tag{3.1.5}$$

as k tends to infinity.

This, and Tomić's result, are related to theorems due to Noble (bibl. 9); and indeed Tomic's method of proving (3.1.4) is similar in some respects to that used by Noble. The point of this and the next section is that a direct application of Noble's method is enough to prove Theorem VIII.

It would be interesting to know whether the term $\log n_k$ in (3.1.5) can be suppressed. If this could be done it would be an improvement, not only on Theorem VIII, but also on Theorem I of (bibl. 6).

§ 3.2. In proving Theorem VIII we assume $x_0 = 0$; there is no loss of generality in this.

Essentially, Noble proved

LEMMA 7. *Let m be a positive integer and* $0 < \delta < \pi$. *Then there exists a trigonometric polynomial*

$$T(x,m,\delta) = 1 + \sum_{j=1}^{m} t_j(m) \cos jx \qquad (3.2.1)$$

such that

$$|T(x,m,\delta)| < A_1 \delta^{-1} \quad (\text{all } x), \qquad (3.2.2)$$

$$\left.\begin{array}{l} |T(x,m,\delta)| < A_2 m^2 \delta^{-1} \exp(-A_3 \delta m) \\ (\delta \leq |x| \leq 2\pi - \delta) \end{array}\right\}, \qquad (3.2.3)$$

where A_1, A_2, A_3 *are positive absolute constants*.

The hypothesis (3.1.2) means that n_k grows so rapidly that the constant $c > 0$ may be chosen so small that, if we let m_k be the integer part of cn_k, then

$$m_k < \min(n_k - n_{k-1},\ n_{k+1} - n_k) \qquad (3.2.4)$$

for all large enough k. Next, put

$$\delta_k = C\, m_k^{-1} \log m_k$$

where the constant C is chosen so large that

$$m_k^2 \delta_k^{-1} \exp(-A_3 \delta_k m_k) = O(1)\, m_k^{-\alpha}; \qquad (3.2.5)$$

such a choice of C is possible because the left-hand side of (3.2.5) is $m_k^{3 - A_3 C}/(C \log m_k)$.

Let $T(x,m_k,\delta_k)$ be as in Lemma 7; then by (3.2.1) and (3.2.4) we have, for all sufficiently large k,

$$a_k \quad = \quad \pi^{-1} \int_{-\pi}^{\pi} (f(x)-f(0))\ T(x,m_k,\delta_k) \cos n_k x \ dx.$$

In the range $-\delta_k < x < \delta_k$ the integrand is

$$O(1)\ \delta_k^{\ \alpha-1}$$

by (3.1.3) (with $x_0 = 0$) and (3.2.2), and so the integral over this range is

$$O(1)\ \delta_k^{\ \alpha}.$$

By (3.2.3) and (3.2.5) the remainder of the integral is

$$O(1)\ m_k^{-\alpha}.$$

Hence by definition of m_k and δ_k we get the estimate (3.1.5) for a_k. Since a similar argument is available for b_k, this proves Theorem VIII.

§ 3.3. In the case $\alpha = 1$ Theorem VIII provides us, of course, with a very wide class of continuous nowhere-differentiable functions. This may be worth looking at more closely, since a weaker method --- making use of a simpler polynomial than Noble's (3.2.1) --- gives a construction which might easily be used in the class-room. In fact, put

$$R(x,m) \quad = \quad \cos^{2m} \tfrac{1}{2}x.$$

It is easy to prove by induction that R takes the form

$$R(x,m) \quad = \quad r_0(m) \quad + \sum_{j=1}^{m} r_j(m) \cos jx;$$

that

$$|R(x,m)| \quad \leqslant \quad 1 \text{ (all } x \text{ and } m),$$

$$|R(x,m)| \quad \leqslant \quad \exp\,(-m\delta^2) \quad (0 < \pi\delta \leqslant |x| \leqslant \pi),$$

and

$$r_j(m) > A_4\ m^{-\frac{1}{2}}$$

where A_4 is a positive absolute constant. The use of R instead of T in our argument, with $\alpha = 1$, leads to the estimate

$$a_k = O(1)\, m_k^{\frac{1}{2}} (\delta_k{}^2 + \exp(-m_k \delta_k{}^2))$$

provided (3.2.4) is true. In particular, if we take

$$n_k = 2^k, \quad m_k = 2^{k-2}, \quad \delta_k{}^2 = k^2\, 2^{2-k},$$

we get

$$a_k = O(1)\, k^2\, 2^{-\frac{1}{2}k}.$$

Thus, for instance, the function

$$f(x) = \sum_{k=1}^{\infty} k^{-2} \cos 2^k x$$

is a continuous nowhere-differentiable function.

§ 3.4. I prove, finally, a theorem on rearrangement of infinite series. Let

$$p_1, p_2, p_3, \ldots \tag{3.4.1}$$

be a (fixed) permutation of the integers 1, 2, 3,..., and let

$$s(n) = \sum_1^n a_k, \qquad \alpha(n) = \sum_1^n a_{p_k} \tag{3.4.2}$$

for a (variable) real or complex series $\sum a_k$. Agnew (bibl. 1) has shown that the existence of lim s(n) implies lim $\alpha(n)$ = lim s(n) if, and only if, the permutation (3.4.1) has the following property: there is an integer N such that for each n = 1,2,3,... the set $(p_1, p_2, \ldots, p_n)$ is the union of at most N blocks of consecutive integers.

It seems now natural to ask the question: are there permutations (3.4.1) such that lim $\alpha(n)$ exists whenever lim s(n) exists (but these limits are perhaps not equal)? The following theorem answers "no" to this question, by showing that if a given rearrangement preserves the convergence of every convergent series, then it preserves the sum of every convergent series.

THEOREM IX. Suppose that there exists a real or complex series $\sum a_k$ such that

$$\lim s(n) = s, \quad \lim \alpha(n) = \alpha \neq s.$$

Then there exists a convergent series $\sum b_k$ such that $\sum b_{p_k}$ is divergent.

In the proof of Theorem IX there is no loss of generality in assuming that the a_k are all real and that $\alpha > s$. Let

$$\alpha - s = 3\delta > 0, \tag{3.4.3}$$

and choose sequences of positive integers N_m, r_m, β_m and γ_m as follows:

$$N_1 = 1; \tag{3.4.4}$$

$$r_1 > 1 \text{ so that } s(r_1) < s + \delta; \tag{3.4.5}$$

$$\beta_1 > r_1 \text{ so that } \alpha(\beta_1) > \alpha - \delta, \tag{3.4.6}$$

$$(1,2,\ldots,r_1) \subset (p_1,p_2,\ldots,p_{\beta_1}); \tag{3.4.7}$$

$$\gamma_1 = \max p_k \text{ for } k = 1, 2, \ldots, \beta_1; \tag{3.4.8}$$

and in general, for $m = 2, 3, \ldots$

$$N_m > \gamma_{m-1} \text{ so that } |s-s(n)| < m^{-3} \text{ for all } n > N_m; \tag{3.4.9}$$

$$r_m > N_m \text{ so that } s(r_m) < s + \delta; \tag{3.4.10}$$

$$\beta_m > r_m \text{ so that } \alpha(\beta_m) > \alpha - \delta, \tag{3.4.11}$$

$$(1,2,\ldots,r_m) \subset (p_1,p_2,\ldots,p_{\beta_m}); \tag{3.4.12}$$

$$\gamma_m = \max p_k \text{ for } k = 1, 2, \ldots, \beta_m. \tag{3.4.13}$$

For $m = 1, 2, 3, \ldots$ put

$$b_k = m a_k \quad (k = N_m, N_m+1, \ldots, N_{m+1}-1). \tag{3.4.14}$$

From (3.4.9) and the General Principle of Convergence it is a simple matter to verify that the series $\sum b_k$ is convergent.

Define $s'(n)$, $\alpha'(n)$ by (3.4.2) with a_k, a_{p_k} replaced by b_k, b_{p_k} respectively. It is clear that (3.4.4), (3.4.5),...,(3.4.13) define sequences of positive integers with the property that, for all m, the set of integers

$$(p_1, p_2, \ldots, p_{\beta_m}) \; - \; (1, 2, \ldots, r_m)$$

is defined and is a subset of the set of integers

$$(N_m, N_m+1, \ldots, N_{m+1}-1).$$

Thus by (3.4.14),

$$\alpha'(\beta_m) - s'(r_m) \quad = \quad m\,(\alpha(\beta_m) - s(r_m)),$$

whence by (3.4.3), (3.4.10) and (3.4.12),

$$\alpha'(\beta_m) - s'(r_m) \; > \; m\delta.$$

Since $\lim s'(r_m)$ exists, it follows at once that $\lim \alpha'(\beta_m) = \infty$, and so $\sum b_{p_k}$ is divergent. This proves Theorem IX.

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